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J0[Phi_] := Puncture[3, 4, {
  Phi~Up~{1, 3, {2, 4}},
  (-Phi)~Up~{3, 2, 4},
  RR[3, 2],
  Phi~Up~{2, 3, 4},
  (-Phi)~Up~{1, 2, {3, 4}}
}];

```

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ToDegree[1, J0[Phi[Vb]]]

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$$\left\{0, 0, \frac{1}{2} \text{har}[2, 3], 0, 0\right\}$$

```

ToDegree[2, J0[Phi[Vb]]]

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$$\left\{-\frac{1}{24} h^2 \text{ar}[2, 3] h[1] + \frac{1}{24} h^2 \text{ar}[1, 3] h[2], 0, \frac{1}{2} \text{har}[2, 3], 0, \right. \\ \left. -\frac{1}{24} h^2 \text{ar}[2, 3] h[1] - \frac{1}{24} h^2 \text{ar}[2, 4] h[1] + \frac{1}{24} h^2 \text{ar}[1, 3] h[2] + \frac{1}{24} h^2 \text{ar}[1, 4] h[2]\right\}$$

```

J1[Phi_] := PCollect[NonCommutativeMultiply @@ J0[Phi]];

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ToDegree[2, J1[Phi[Vb]]]

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$$-\frac{1}{24} h^2 \text{ar}[2, 4] h[1] + \text{ar}[2, 3] \left(\frac{h}{2} - \frac{1}{12} h^2 h[1]\right) + \frac{1}{12} h^2 \text{ar}[1, 3] h[2] + \frac{1}{24} h^2 \text{ar}[1, 4] h[2]$$

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ToDegree[4, J0[Phi[Vb]]]

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$$\left\{\text{ar}[2, 3] \left(-\frac{1}{24} h^2 h[1] + \frac{11 h^4 h[1]^3}{15360} + \frac{67 h^4 h[1]^2 h[2]}{138240} + \frac{169 h^4 h[1] h[2]^2}{138240}\right) + \right. \\ \text{ar}[1, 3] \left(\frac{1}{24} h^2 h[2] - \frac{169 h^4 h[1]^2 h[2]}{138240} - \frac{67 h^4 h[1] h[2]^2}{138240} - \frac{11 h^4 h[2]^3}{15360}\right), 0, \frac{1}{2} \text{har}[2, 3], \\ 0, \text{ar}[2, 3] \left(-\frac{1}{24} h^2 h[1] + \frac{31 h^4 h[1]^3}{46080} + \frac{23 h^4 h[1]^2 h[2]}{46080} + \frac{13 h^4 h[1] h[2]^2}{46080}\right) + \\ \text{ar}[2, 4] \left(-\frac{1}{24} h^2 h[1] + \frac{31 h^4 h[1]^3}{46080} + \frac{23 h^4 h[1]^2 h[2]}{46080} + \frac{13 h^4 h[1] h[2]^2}{46080}\right) + \\ \text{ar}[1, 3] \left(\frac{1}{24} h^2 h[2] - \frac{19 h^4 h[1]^2 h[2]}{17280} - \frac{7 h^4 h[1] h[2]^2}{4320} - \frac{7 h^4 h[2]^3}{5760}\right) + \\ \left. \text{ar}[1, 4] \left(\frac{1}{24} h^2 h[2] - \frac{19 h^4 h[1]^2 h[2]}{17280} - \frac{7 h^4 h[1] h[2]^2}{4320} - \frac{7 h^4 h[2]^3}{5760}\right)\right\}$$

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ToDegree[4, J1[Phi[Vb]]]

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$$\text{ar}[2, 3] \left(\frac{h}{2} - \frac{1}{12} h^2 h[1] + \frac{1}{720} h^4 h[1]^3 + \frac{17 h^4 h[1]^2 h[2]}{17280} - \frac{h^4 h[1] h[2]^2}{4320}\right) + \\ \text{ar}[2, 4] \left(-\frac{1}{24} h^2 h[1] + \frac{31 h^4 h[1]^3}{46080} + \frac{23 h^4 h[1]^2 h[2]}{46080} + \frac{13 h^4 h[1] h[2]^2}{46080}\right) + \\ \text{ar}[1, 4] \left(\frac{1}{24} h^2 h[2] - \frac{19 h^4 h[1]^2 h[2]}{17280} - \frac{7 h^4 h[1] h[2]^2}{4320} - \frac{7 h^4 h[2]^3}{5760}\right) + \\ \text{ar}[1, 3] \left(\frac{1}{12} h^2 h[2] - \frac{107 h^4 h[1]^2 h[2]}{46080} - \frac{97 h^4 h[1] h[2]^2}{46080} - \frac{h^4 h[2]^3}{5120}\right)$$

```
J2[Phi_] := J1[Phi] /. PCollect[ar[i_, j_] -> ar[i, j - 2]];
ToDegree[1, J2[Phi[Vb]]]
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$$\frac{1}{2} h \operatorname{ar}[2, 1]$$

```
V2 = ToDegree[2, J2[Phi[Vb]]]
```

$$-\frac{1}{24} h^2 \operatorname{ar}[2, 2] h[1] + \operatorname{ar}[2, 1] \left(\frac{h}{2} - \frac{1}{12} h^2 h[1] \right) + \frac{1}{12} h^2 \operatorname{ar}[1, 1] h[2] + \frac{1}{24} h^2 \operatorname{ar}[1, 2] h[2]$$

```
ToDegree[2, {R4Eqn[V2], TwistEqn[V2]}]
```

$$\left\{ -h^2 \operatorname{ar}[2, 3] h[1] + h^2 \operatorname{ar}[1, 3] h[2], h \operatorname{ar}[1, 2] - h \operatorname{ar}[2, 1] - \frac{1}{4} h^2 \operatorname{ar}[2, 1] h[1] - \frac{1}{4} h^2 \operatorname{ar}[2, 2] h[1] + \frac{1}{4} h^2 \operatorname{ar}[1, 1] h[2] + \frac{1}{4} h^2 \operatorname{ar}[1, 2] h[2] \right\}$$

```
ToDegree[2, {R4Eqn[V2~Up~{2, 1}], TwistEqn[V2~Up~{2, 1}]]
```

$$\left\{ 0, -\frac{1}{4} h^2 \operatorname{ar}[2, 1] h[1] - \frac{1}{4} h^2 \operatorname{ar}[2, 2] h[1] + \frac{1}{4} h^2 \operatorname{ar}[1, 1] h[2] + \frac{1}{4} h^2 \operatorname{ar}[1, 2] h[2] \right\}$$

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PCollect[ToDegree[1, Vb]]
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$$\frac{1}{4} h \operatorname{ar}[1, 2] - \frac{1}{4} h \operatorname{ar}[2, 1]$$

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PCollect[ToDegree[2, Vb]]
```

$$\frac{1}{96} h^2 \operatorname{ar}[2, 2] h[1] + \frac{1}{96} h^2 \operatorname{ar}[1, 1] h[2] + \operatorname{ar}[2, 1] \left(-\frac{h}{4} + \frac{5}{96} h^2 h[1] + \frac{1}{32} h^2 h[2] \right) + \operatorname{ar}[1, 2] \left(\frac{h}{4} + \frac{1}{32} h^2 h[1] + \frac{5}{96} h^2 h[2] \right)$$

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ToDegree[4, J2[Phi[Vb]]]
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$$\begin{aligned} & \operatorname{ar}[2, 1] \left(\frac{h}{2} - \frac{1}{12} h^2 h[1] + \frac{1}{720} h^4 h[1]^3 + \frac{17 h^4 h[1]^2 h[2]}{17280} - \frac{h^4 h[1] h[2]^2}{4320} \right) + \\ & \operatorname{ar}[2, 2] \left(-\frac{1}{24} h^2 h[1] + \frac{31 h^4 h[1]^3}{46080} + \frac{23 h^4 h[1]^2 h[2]}{46080} + \frac{13 h^4 h[1] h[2]^2}{46080} \right) + \\ & \operatorname{ar}[1, 2] \left(\frac{1}{24} h^2 h[2] - \frac{19 h^4 h[1]^2 h[2]}{17280} - \frac{7 h^4 h[1] h[2]^2}{4320} - \frac{7 h^4 h[2]^3}{5760} \right) + \\ & \operatorname{ar}[1, 1] \left(\frac{1}{12} h^2 h[2] - \frac{107 h^4 h[1]^2 h[2]}{46080} - \frac{97 h^4 h[1] h[2]^2}{46080} - \frac{h^4 h[2]^3}{5120} \right) \end{aligned}$$

```
PCollect[ToDegree[4, Vb]]
```

$$\begin{aligned} & \frac{1}{96} h^2 \operatorname{ar}[2, 2] h[1] + \frac{1}{96} h^2 \operatorname{ar}[1, 1] h[2] + \\ & \operatorname{ar}[1, 2] \left(\frac{h}{4} + \frac{1}{32} h^2 h[1] - \frac{19 h^4 h[1]^3}{46080} + \frac{5}{96} h^2 h[2] - \frac{169 h^4 h[1]^2 h[2]}{138240} - \frac{187 h^4 h[1] h[2]^2}{138240} - \frac{11 h^4 h[2]^3}{15360} \right) + \\ & \operatorname{ar}[2, 1] \left(-\frac{h}{4} + \frac{5}{96} h^2 h[1] - \frac{11 h^4 h[1]^3}{15360} + \frac{1}{32} h^2 h[2] - \frac{187 h^4 h[1]^2 h[2]}{138240} - \frac{169 h^4 h[1] h[2]^2}{138240} - \frac{19 h^4 h[2]^3}{46080} \right) \end{aligned}$$